Bialgebras are Algebras, and Drinfeld doubles are Centers

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Summary

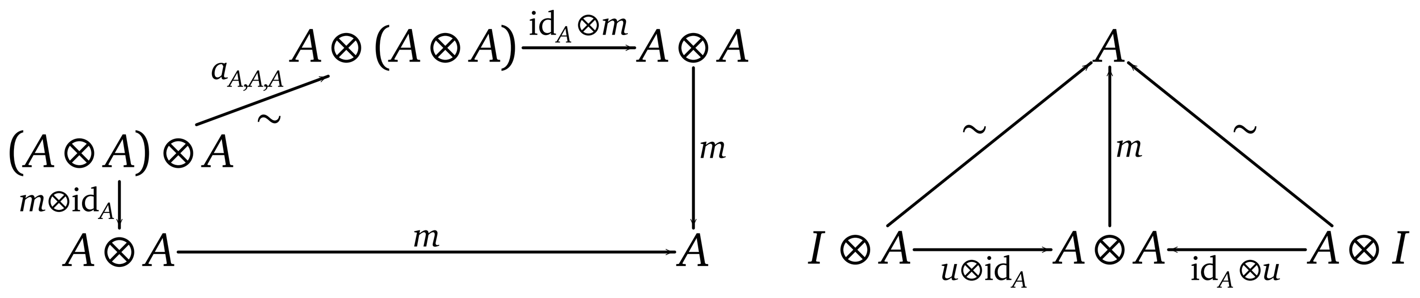
Algebras in a monoidal category is a concise way of referring to an object which has unital and associative multiplicative structures. For e.g., monoids, rings, algebras over a field, monads, operads are all algebras once you choose the right monoidal category. In some occasions the language of algebras in a monoidal 1-category is not enough, and one can simply categorify it to obtain the language of 2-algebras (also called psuedomonoids by Day and Street) in a monoidal 2-category to encompass more examples. For instance, monoidal categories are hard to formulate as an algebra in a monoidal category. However, they're precisely 2-algebras in the monoidal 2-category of all categories. Recently, this view has also been emphasized by Thibault who proved that many notions in tensor category theory arise as certain kinds of 2-algebras in a monoidal 2-category. One should note that once an object $\mathcal{X}$ is formulated as an algebra or a 2-algebra, questions related to this new viewpoint emerge. For instance, one can compute the *center* of $\mathcal{X}$. In our work, we study this problem when $\mathcal{X}$ is a quasi-bialgebra, which is observed to be a 2-algebra in certain symmetric monoidal 2-category $\mathscr{A}$ by McCrudden. To this end we first define center of an algebra in a symmetric monoidal 2-category, which is modified from the one given by Lurie in another setting. We show that when the quasi-bialgebra $\mathcal{X}$ is finite-dimensional Hopf, then the center of $\mathcal{X}$ coincides with the Drinfeld double of $\mathcal{X}$. This echos with the idea that "each mathematical construction is universal". The poster surveys McCrudden's idea and introduces this part of our work.
our work is still on-going, and we are also interested in computing the center of quasi-Hopf algebras, as well as augmented algebras and quasi-triangular quasi-bialgebras, with the latter two corresponding to E_0-2-algebras and braided 2-algebras (also called braided pseudomonoids) in $\mathscr{A}$ respectively. We will also prove a relation between 2-algebras and braided 2-algebras, which is an important ingredient in the basic theory of centers.

§1 Algebras in a Monoidal Category and 2-Algebras in a Monoidal 2-Category.

Definition. A *monoidal category* is a category C equipped with a functor $\otimes: C \times C \to C$ called the tensor product, a distingshed object $I \in C$ called the tensor unit, a natural isomorphism $\{a_{X,Y,Z}: (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)\}_{X,Y,Z\in C}$ and some additional structures satisfying certain coherence conditions.

Example. (1) The category **Set** of sets and maps is a monoidal category with $\otimes = \times$ and $I = \{*\}$. The natural isomorphisms $a_{X,Y,Z}: (X\times Y)\times Z \xrightarrow{\sim} X\times(Y\times Z)$ is canonicaolly determined. (2) Fix an algebraic closed field $\Bbbk$ of characteristic 0. The category **Alg** of finite dimensional $\Bbbk$-algebras and algbra homomorphisms is a monoidal category with $\otimes := \otimes_\Bbbk$ and $I = \Bbbk$. (3) Given a monoidal category C, we use $C^{1-\mathrm{op}}$ to denote the monoidal category obtained from C by reversing all 1-morphisms.

Definition. An *algebra* in a monoidal category $(C,\otimes,I)$ is a triple $(A, m: A\otimes A \to A, u: I \to A)$ where $A \in C$ is an object and m and u are morphisms in C rendering the following diagrams commute:

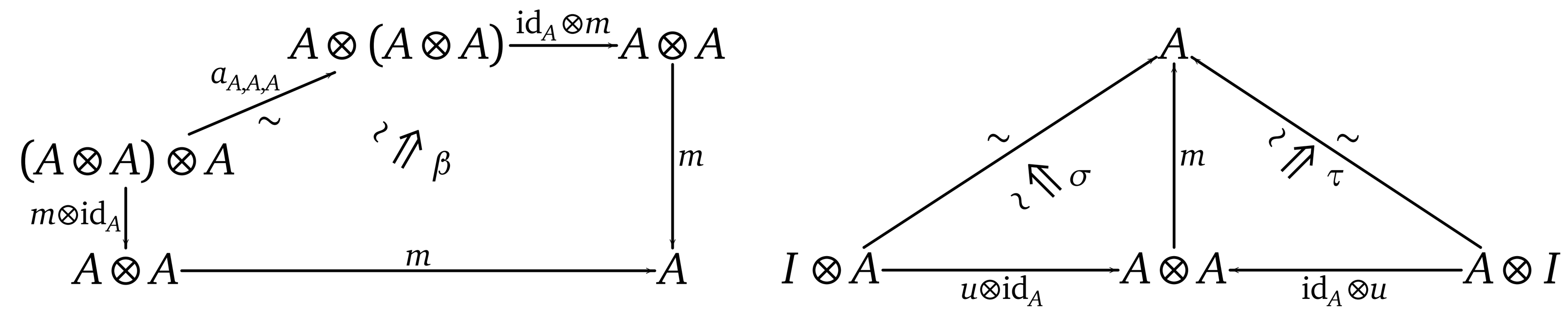


Example. (1) An algebra (A,m,u) in **Set** is a monoid, whose multiplication is given by m and the unit is given by $u(*) \in A$. (2) An algebra (A,Δ,ϵ) in $\mathbf{Alg}^{1-\mathrm{op}}$ is precisely a finite dimensional bialgebra. Indeed, a f.d. bialgebra is a f.d. $\Bbbk$-algebra $A = (A, M: A\otimes A \to A, U: \Bbbk \to A)$ equipped with algebra homomorphisms $\Delta: A \to A\otimes A$ and $\epsilon: A \to \Bbbk$ satisfying coassociativity and counitality.

While $\mathbf{Alg}^{1-\mathrm{op}}$ is a playground simple enough for defining bialgebras, it is not a very effective one for two reasons: (a) it does not capture quasi-bialgebras; (b) it does not capture the intricate structures of an important class of bialgebras: quasi-triangular bialgebras. This leads us to McCrudden's monoidal 2-category $\mathscr{A}$ and the 2-algebras in it. We begin by the definition of monoidal 2-categories and 2-algebras.

Definition. A *monoidal 2-category* is a 2-category C equipped with a 2-functor $\otimes: C\times C \to C$, a distingshed object $I \in C$ serving as the unit, a famly of invertible natural 1-morphisms $\{a_{X,Y,Z}: (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)\}_{X,Y,Z\in C}$ and many additional structures satisfying some coherence relations.

Definition. A *2-algebra* in a monoidal 2-category $(C,\otimes,I,a)$ is a sixtuple $(A, m: A\otimes A\to A, u: I \to A, \beta,\sigma,\tau)$ where $A \in C$ is an object, m,u are 1-morphisms in C, and β,σ,τ are invertible 2-morphisms with depicted in the figures below respectively, satisfying certain relations.



If the 2-morphisms β,σ,τ are identities, we say A is *strict*.

Example. (1) 2-algebras in the monoidal 2-category **Cat** of all categories are monoidal categories.

(2) The monoidal 2-category $\mathscr{A}$. As a 2-category, it has:

- objects being finite dimensional unital associative $\Bbbk$-algebras;
- reverse 1-morphisms being algebra homomorphisms, i.e., a 1-morphism $A \to B$ is an algebra homomorphism $B \to A$;
- 2-morphisms being *intertwinners* bewteen algebra homomorphisms, where an interwinner $b: \phi \Rightarrow \psi$ for algebra homomorphisms $\phi,\psi: A \to B$ is an element $b \in B$ such that $b\phi(a) = \psi(a)b, \forall a \in A$.

The monoidal 2-structure of $\mathscr{A}$ is given by the tensor product $\otimes_\Bbbk$ of vector spaces.

(3) A 2-algebra in $\mathscr{A}$ is by definition a sixtuple $(A,\Delta,\epsilon,\beta,\sigma,\tau)$ where $A = (A,m,u)$ is a f.d. $\Bbbk$-algebra, $\Delta: A \to A\otimes A$ and $\epsilon: A \to \Bbbk$ are algebra homomorphisms, β is an invertible element $\beta \in A\otimes A\otimes A$ such that $\beta(\Delta\otimes \mathrm{id}_A)(\Delta(a)) = (\mathrm{id}_A\otimes\Delta)(\Delta(a))\beta$ for all $\beta \in A$, and σ,τ are similiar interwinners related to ϵ. Thus 2-algebras in $\mathscr{A}$ are precisely quasi-bialgebras, while the strict ones correspond to bialgebras.

One can furthermore check that another two kinds of 2-algebras, braided/sylleptic strict 2-algebras in $\mathscr{A}$, are precisely in correspondence with quasi-triangular/triangular bialgebras. Above are all encompassed by McCrudden's 2002 paper.

§2 Centers of 1- and 2-algebras

Suppose now C is braided.

Definition. The *center* of an algebra $A \in C$ is a pair $(Z(A), \rho: Z(A)\otimes A \to A)$ where $Z(A)$ is an algebra in C, and ρ is an algebra homomorphism in C satisfying $\rho\circ(u_{Z(A)}\otimes \mathrm{id}_A) = \mathrm{id}_A$, such that it is terminal among all such pairs.

Example. When C is the category **Vec** of $\Bbbk$-vector spaces, the center of an algebra $A \in C$ recovers the usual center of a $\Bbbk$-algebra as the first argument $Z(A)$, while the universal algebra homomorphism $\rho: Z(A)\otimes_\Bbbk A \to A$ is given by multiplication $z\otimes a \mapsto za$.

Using the language of reprentable 2-functors, we routinely categorifies this notion to obtain the definition of center of an algebra in a braided monoidal 2-category. It is worthwhile to note here that a 2-algebra homomorphism in a monoidal 2-category is not only a 1-morphism satisfying *extra conditions*, but rather a 1-morphism equipped with two invertible 2-morphisms as *extra data* satisfying certain conditions.

Selected Main Theorem

The center of a finite dimensional Hopf algebra (H,Δ,ϵ) is its Drinfeld double $D(H)$, where the universal homomorphism of quasi-bialgebras $H \to D(H)\otimes H$ is given by $h \mapsto \epsilon\otimes h_{(1)}\otimes h_{(2)}$ equipped two invertible elements $1_H \in H$ and $\sum_i \epsilon\otimes 1_H\otimes e_i\otimes e^i\otimes 1_H\otimes 1_H \in D(H)\otimes H\otimes D(H)\otimes H$. Here we have identified $D(H) \cong H^*\otimes H$ as a vector space, and $\{e_i\}_i$ is a basis of H.

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